MATH4210 Tutorial 10

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that, for some constant $C > 0$, $f(x) \leq e^{C|x|}$ for all $x \in \mathbb{R}$. Define

$$u(t, x) := E[f(B_T)|B_t = x] = E[f(B_T - B_t + x)].$$

Prove that

(a) $\partial_x u(t, x) = E[\frac{B_T - B_t}{T-t} f(x + B_T - B_t)]$.

(b) $\partial^2_{xx} u(t, x) = E[\frac{(B_T - B_t)^2 - (T-t)}{(T-t)^2} f(x + B_T - B_t)]$.

(c) $\cancel{\partial_t u(t, x) = \int_{\mathbb{R}} f(x + y)\partial_t \rho(t, y)\, dy}$. $\partial_t u(t,x) = -\frac{1}{2}\partial^2_{xx} u(t,x)$.

(d) Apply Itô formula on $u(t, B_t)$, deduce that u satisfies the heat equation

$$\partial_t u + \frac{1}{2}\partial^2_{xx} u = 0.$$

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(a) $\partial_x u(t,x) = E[\frac{B_T - B_t}{T-t} f(x + B_T - B_t)]$.

Solution:

$$u(t,x) = E[f(B_T - B_t + x)] \qquad Z = B_T - B_t \sim N(0, T-t)$$

$$= \int_{\mathbb{R}} f(z+x) \cdot f_Z(z)\, dz.$$

$$\partial_x u(t,x) = \int_{\mathbb{R}} f'(z+x) \cdot f_Z(z)\, dz$$

$$= \int_{\mathbb{R}} f_Z(z)\, d f(z+x).$$

integration by parts $= f_Z(z) \cdot f(z+x) \Big|_{z=-\infty}^{z=+\infty} - \int_{\mathbb{R}} f(z+x)\, d f_Z(z)$

$$= \int_{\mathbb{R}} f(z+x) \cdot \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{z^2}{2(T-t)}} \cdot \frac{z}{T-t}\, dz$$

$$= E[f(x + B_T - B_t) \cdot \frac{B_T - B_t}{T-t}].$$

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Solution: $\partial_x u(t,x) = \int_{\mathbb{R}} \frac{z}{T-t} f(x+z) \cdot \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{z^2}{2(T-t)}} dz.$

$$\partial^2_{xx} u(t,x) = \int_{\mathbb{R}} \frac{z}{T-t} \cdot f'(x+z) \cdot \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{z^2}{2(T-t)}} dz$$

$$= \int_{\mathbb{R}} \left[\frac{z}{T-t} \cdot \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{z^2}{2(T-t)}}\right] d f(x+z)$$

integration by parts $= f(x+z) \frac{z}{T-t} \cdot f_Z(z) \Big|_{z=-\infty}^{z=+\infty}$

$$- \int_{\mathbb{R}} f(x+z)\, d\left(\frac{z}{T-t} f_Z(z)\right).$$

$$= -\int_{\mathbb{R}} f(x+z) \cdot \frac{1}{T-t} \cdot \left(f_Z(z) - \frac{z^2 f(z)}{T-t} \right) dz$$

$$= \int_{\mathbb{R}} f(x+z) \cdot \frac{z^2 - (T-t)}{(T-t)^2} \cdot f_Z(z) \cdot dz$$

$$= E\left[f(x+B_T-B_t) \cdot \frac{(B_T-B_t)^2-(T-t)}{(T-t)^2} \right]$$

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Solution: $$u(t,x) = \int_{\mathbb{R}} f(x+z)\cdot f_Z(z)\,dz.$$

$$= \int_{\mathbb{R}} f(x+z)\cdot \underbrace{\frac{1}{\sqrt{2\pi(T-t)}}\, e^{-\frac{z^2}{2(T-t)}}}_{f_Z(z)}\,dz$$

$$\partial_t u(t,x) = \int_{\mathbb{R}} f(x+z)\cdot \partial_t f_Z(z)\,dz.$$

$$= \int_{\mathbb{R}} f(x+z)\cdot f_Z(z)\cdot \frac{1}{2}\cdot\left(\frac{(T-t)-z^2}{(T-t)^2}\right)dz.$$

$$= \frac{1}{2}\cdot \int_{\mathbb{R}} f(x+z)\cdot f_Z(z)\cdot \frac{z^2-(T-t)}{(T-t)^2}\,dz$$

$$= -\frac{1}{2} E\left[f(x+B_T-B_t)\cdot \frac{(B_T-B_t)^2-(T-t)}{(T-t)^2}\right].$$

$$\underline{\partial_t u + \frac{1}{2}\partial^2_{xx} u(t,x) = 0.}$$

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$u(t, B_t) = E[f(B_T)|B_t]$.

(d) Apply Itô formula on $u(t, B_t)$, deduce that u satisfies the heat equation

$$\partial_t u + \frac{1}{2}\partial^2_{xx} u = 0.$$

Solution: $u(t+h, B_{t+h}) = u(t, B_t) + \int_t^{t+h} (\partial_t u + \frac{1}{2}\partial^2_{xx} u)(s, B_s)\, ds$

$\int_t^{t+h} \partial_x u(s, B_s)\, dB_s$

Hint: $E[u(t+h, B_{t+h}) | B_t = x] = E[f(B_T)|B_t = x]$.

$E[u(t+h, B_{t+h}) | B_t = x] = u(t,x)$.

$E[u(t, B_t) | B_t = x] = u(t,x)$

$E[\int_t^{t+h} \partial_x u\, dB_s | B_t = x] = 0$.

$E[\frac{1}{h}\int_t^{t+h} (\partial_t u + \frac{1}{2}\partial^2_{xx} u)(s, B_s)\, ds \,|B_t = x] = 0$.

Let $h \to 0$, $E[(\partial_t u + \frac{1}{2}\partial^2_{xx} u)(t, B_t) | B_t = x] = 0$.

$\Rightarrow \partial_t u + \frac{1}{2}\partial^2_{xx} u(t,x) = 0$.